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Generalized Lorentz invariance and solitary waves for coupled nonlinear Klein–Gordon equations

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Abstract

A generalized Lorentz transformation that preserves the invariance of a nonlinear Klein–Gordon equation is presented and is used to extend the set of solitary-wave solutions for a special set of *N* coupled nonlinear Schrödinger equations to become solutions of the corresponding set of coupled nonlinear Klein–Gordon equations.

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1. Introduction

Solitary-wave solutions for a one-component nonlinear Klein–Gordon (NKG) and two-component coupled nonlinear Klein–Gordon (CNKG) equations that are an extension of the corresponding nonlinear Schrödinger (NLS) and coupled nonlinear Schrödinger (CNLS) equations were recently given by the author [1]. The NLS and CNLS equations have been studied extensively for many years (see [2–5] and many references therein) because of their useful applications for long-distance propagation in optical communication systems. If, however, the wavefunctions that represent the envelopes of the waves are not as slowly varying functions of the position as are usually assumed, the corresponding nonlinear Klein–Gordon type equations instead of the nonlinear Schrödinger equations would be the equations that govern their evolution.

In an attempt by the author to extend the applicability of the large number of periodic solitary-wave solutions [6] that are available for a special set of the CNLS equations to the corresponding set of the CNKG equations, *two* forms of a generalized Lorentz transformation were discovered and they are presented in section 2. The extension of our previous results [1, 6] on solitary waves for a special set of CNLS equations to the corresponding sets of *N* CNKG equations is presented in section 3. The two corresponding sets will be called the L-sets as they are a direct extension of the L-set for *N* CNLS equations we studied earlier [6] for which we found solitary waves that can be represented by Lamé functions [7] of orders $n \leqslant N$. As it will be seen, the two forms of solutions given by the two forms of our generalized Lorentz transformation can cover the same regime of parameter values, in which they would give equivalently the same solutions, or they can be two separate solutions for two different regimes of parameter values. A brief summary is given in section 4.

2. Generalized Lorentz transformation

Consider the following one-component nonlinear general Klein–Gordon equation for the complex amplitude or wavefunction $\phi(z,t)$ as a function of position z and time t:

$$\mathrm{i}\alpha'\phi_z + \alpha''\phi_{zz} + \mathrm{i}\beta'\phi_t + \beta''\phi_{tt} + \kappa\phi + F(|\phi|^2)\phi = 0 \tag{1}$$

where α', α'', β' and β'' are real parameters, κ can be real or complex in this section but is assumed real in the following section, $F(|\phi|^2)$ is an arbitrary function of $|\phi|^2$, and where the subscripts in z and t denote derivatives with respect to z and t.

For the case $\alpha'' \neq 0$ and $\beta'' \neq 0$, the first derivative terms with respect to z and t can be eliminated with the substitution

$$\phi(z,t) = \psi(z,t)\exp\{-\mathrm{i}(\alpha' z/\alpha'' + \beta' t/\beta'')/2\}$$

which transforms equation (1) into the following 'standard form' of nonlinear Klein–Gordon equation for ψ:

$$\alpha''\psi_{zz} + \beta''\psi_{tt} + \mu\psi + F(|\psi|^2)\psi = 0 \tag{2}$$

with $\mu = \kappa + \alpha'^2/(4\alpha'') + \beta'^2/(4\beta'')$.

However, in order to include the special cases of two forms of nonlinear Schrödinger equations given by (i) $\alpha'' = 0$, $\beta'' \neq 0$, $\alpha' \neq 0$, and (ii) $\beta'' = 0$, $\alpha'' \neq 0$, $\beta' \neq 0$, the general nonlinear Klein–Gordon equation (1) is used as the starting point. While for the standard Klein–Gordon equation, the ratio $\alpha''/\beta'' = -c^2$ is fixed, where c is the speed of light, for the example of pulse propagation in an optical fibre, the parameters α', β', α'' and β'' in equation (1) depend on the characteristics of the medium, and the ratio α''/β'' depends on the group velocity of the wave and its dispersion and can have a positive or negative value.

These considerations led to the formulation of a generalized Lorentz transformation which we present in the following.

Consider changing the position coordinate and time from z and t to z' and t' according to the following transformation:

$$z' = a_{11}z + a_{12}t \qquad t' = a_{21}z + a_{22}t \tag{3}$$

and

$$z = a_{11}z' - a_{12}t' \qquad t = -a_{21}z' + a_{22}t' \tag{4}$$

where a are constants to be determined. Consider a function $\phi'(z',t') = \phi(z,t)\exp[-\mathrm{i}\theta(z', t')]$ or

$$\phi(z,t) = \phi'(z',t')\exp[\mathrm{i}\theta(z',t')]. \tag{5}$$

Substituting equation (4) into equation (1), and using $\partial/\partial z = a_{11}\partial/\partial z' + a_{21}\partial/\partial t'$, and $\partial/\partial t = a_{12}\partial/\partial z' + a_{22}\partial/\partial t'$ equation (1) becomes

$$\begin{aligned}
&\mathrm{i}\alpha'\phi_z + \alpha''\phi_{zz} + \mathrm{i}\beta'\phi_t + \beta''\phi_{tt} + \kappa\phi + F(|\phi|^2)\phi = \exp[\mathrm{i}\theta(z',t')] \\
&\quad\times \left\{\mathrm{i}\phi'_{z'}\left[2\left(a_{11}^2\alpha'' + a_{12}^2\beta''\right)\theta_{z'} + 2(a_{11}a_{21}\alpha'' + a_{12}a_{22}\beta'')\theta_{t'} + (a_{11}\alpha' + a_{12}\beta')\right]\right. \\
&\quad + \mathrm{i}\phi'_{t'}\left[2\left(a_{21}^2\alpha'' + a_{22}^2\beta''\right)\theta_{t'} + 2(a_{11}a_{21}\alpha'' + a_{12}a_{22}\beta'')\theta_{z'} + (a_{21}\alpha' + a_{22}\beta')\right] \\
&\quad + \left(a_{11}^2\alpha'' + a_{12}^2\beta''\right)\phi'_{z'z'} + \left(a_{21}^2\alpha'' + a_{22}^2\beta''\right)\phi'_{t't'} + 2(a_{11}a_{21}\alpha'' + a_{12}a_{22}\beta'')\phi'_{z't'}
\end{aligned}$$

$$\begin{aligned}
&-\left[\left(a_{11}^2\alpha''+a_{12}^2\beta''\right)\theta_{z'}^2+\left(a_{21}^2\alpha''+a_{22}^2\beta''\right)\theta_{t'}^2+2(a_{11}a_{21}\alpha''+a_{12}a_{22}\beta'')\theta_{z'}\theta_{t'}\right.\\
&+\left(a_{11}^2\alpha''+a_{12}^2\beta''\right)\theta_{z'z'}+\left(a_{21}^2\alpha''+a_{22}^2\beta''\right)\theta_{t't'}+2(a_{11}a_{21}\alpha''+a_{12}a_{22}\beta'')\theta_{z't'}\\
&\left.+(a_{11}\alpha'+a_{12}\beta')\theta_{z'}+(a_{21}\alpha'+a_{22}\beta')\theta_{t'}\right]\phi'+\kappa\phi'+F(|\phi'|^2)\phi'\big\}\\
=\;&\exp[\mathrm{i}\theta(z',t')]\{\mathrm{i}\alpha'\phi'_{z'}+\alpha''\phi'_{z'z'}+\mathrm{i}\beta'\phi'_{t'}+\beta''\phi'_{t't'}+\kappa\phi'+F(|\phi'|^2)\phi'\}=0
\end{aligned}$$

i.e. equation (1) would be invariant with respect to the coordinate and time transformation given by equations (3) and (4), provided that the following relations:

$$a_{11}=\pm a_{22} \qquad a_{11}a_{22}-a_{12}a_{21}=\pm 1 \qquad \alpha''a_{21}\pm\beta''a_{12}=0 \tag{6}$$

are satisfied, and with A and B in $\theta(z',t')=Az'+Bt'$ satisfying the following relations:

for $\alpha''\neq 0$

$$A=\frac{1}{2\alpha''}[(1-a_{11})\alpha'-a_{12}\beta'] \tag{7}$$

for $\beta''\neq 0$

$$B=\frac{1}{2\beta''}[-a_{21}\alpha'+(1-a_{22})\beta'] \tag{8}$$

and generally

$$\alpha''A^2+\beta''B^2+(a_{11}\alpha'+a_{12}\beta')A+(a_{21}\alpha'+a_{22}\beta')B=0. \tag{9}$$

Relation (9) allows A (or B) to be determined for the case α'' (or β'') equal to zero. The coordinate and time transformations given by equations (3) and (4), with a given by equation (6), will be called the generalized Lorentz transformation.

The following result follows: any solution $\phi(z,t)$ of equation (1) can be replaced by

$$\phi(a_{11}z-a_{12}t,-a_{21}z+a_{22}t)\exp[-\mathrm{i}\theta(z,t)] \tag{10}$$

where

$$\theta(z,t)=Az+Bt \tag{11}$$

A and B are given by equations (7) and (8), and where a satisfy equation (6). The invariance will be called the generalized Lorentz invariance.

From here on, we shall consider and apply only the *proper* Lorentz transformation (equations (3) and (4)) and invariance (equations (10) and (11)) for which the a satisfy equation (6) given by the *upper* signs and where $a_{11}=a_{22}=$ a *positive* quantity. We shall drop the description 'proper' for brevity. As can be seen, the generalized Lorentz transformation depends on the ratio α''/β'' or β''/α''. Define

$$r\equiv\alpha''/\beta'' \tag{12}$$

and introduce a velocity parameter v.

We now present two forms of this transformation:

(I) For $r>-v^2$. Setting $a_{11}=a_{22}=\eta\equiv(1+rv^{-2})^{-1/2}$, $a_{12}=-\eta rv^{-1}$, $a_{21}=\eta/v$, we have the following generalized Lorentz transformation:

$$z'=\eta(z-rv^{-1}t) \qquad t'=\eta(t+z/v) \tag{13}$$

or

$$z=\eta(z'+rv^{-1}t') \qquad t=\eta(t'-z'/v).$$

The generalized Lorentz invariance states that any solution $\phi(z,t)$ of equation (1) can be replaced by

$$\phi[\eta(z+rv^{-1}t),\eta(t-z/v)]\exp -\mathrm{i}\left\{\frac{1}{2\alpha''}[(1-\eta)\alpha'+\eta rv^{-1}\beta']z\right.$$
$$\left.+\frac{1}{2\beta''}[-\eta v^{-1}\alpha'+(1-\eta)\beta']t\right\}.$$

(II) For $r^{-1}\geqslant 0$, or for $r<-v^2$. Setting $a_{11}=a_{22}=\sigma\equiv(1+r^{-1}v^2)^{-1/2}$, $a_{12}=\sigma v$, $a_{21}=-\sigma r^{-1}v$, we have the following generalized Lorentz transformation:

$$z'=\sigma(z+vt)\qquad t'=\sigma(t-r^{-1}vz)\tag{14}$$

or

$$z=\sigma(z'-vt')\qquad t=\sigma(t'+r^{-1}vz').$$

The generalized Lorentz invariance states that any solution $\phi(z,t)$ of equation (1) can be replaced by

$$\phi[\sigma(z-vt),\sigma(t+r^{-1}vz)]\exp -\mathrm{i}\left\{\frac{1}{2\alpha''}[(1-\sigma)\alpha'-\sigma v\beta']z\right.$$
$$\left.+\frac{1}{2\beta''}[\sigma r^{-1}v\alpha'+(1-\sigma)\beta']t\right\}.$$

Note that the two forms of generalized Lorentz transformation have an overlapping region of applicability given by $r\geqslant 0$ in which they should give the same result for the same value of r. However, for negative values of r, the two forms are applied to two separate regions given by $r>-v^2$ and $r<-v^2$, respectively.

The familiar Lorentz transformation for special relativity corresponds to the special value of $r=-c^2<-v^2$ for which the second form of our generalized Lorentz transformation applies, giving

$$\sigma=(1-v^2/c^2)^{-1/2}\tag{15}$$

and

$$z'=\sigma(z+vt)\qquad t'=\sigma(t+vz/c^2)\tag{16}$$

or

$$z=\sigma(z'-vt')\qquad t=\sigma(t'-vz'/c^2).$$

It is somewhat unexpected that the first form of the generalized Lorentz transformation can be used, with the choice of $r=-(v^2/c^2)v^2>-v^2$, to give a second and new 'Lorentz' transformation that would have the same 'relativistic' factor $\eta=\sigma=(1-v^2/c^2)^{-1/2}$, for which the transformation becomes

$$z'=\eta(z+v^3t/c^2)\qquad t'=\eta(t+z/v)\tag{17}$$

or

$$z=\eta(z'-v^3t'/c^2)\qquad t=\eta(t'-z'/v).$$

Transformations given by equations (13) and (14) become, for special cases of $r=0$ and $r^{-1}=0$, two forms of Galilean transformation that are applicable for two forms of NLS equations, as shown in the following.

(i) For $r=0$ or $\alpha''=0$ and $\alpha'\neq 0$. Setting $a_{11}=a_{22}=1$, $a_{12}=0$, $a_{21}=1/v$, the following Galilean transformation results:

$$z'=z\qquad t'=t+z/v\tag{18}$$

or

$$z = z' \qquad t = t' - z'/v.$$

The Galilean invariance states that any solution $\phi(z, t)$ of

$$\mathrm{i}\alpha'\phi_z + \mathrm{i}\beta'\phi_t + \beta''\phi_{tt} + \kappa\phi + F(|\phi|^2)\phi = 0$$

can be replaced by

$$\phi(z, t - z/v) \exp\left\{\mathrm{i}\frac{\alpha'}{2\beta''v}\left[t - \left(\frac{1}{2v} + \frac{\beta'}{\alpha'}\right)z\right]\right\}$$

where B and A for θ have been obtained from equations (8) and (9). This form of Galilean invariance was used often in problems in nonlinear optics (see e.g. [2–5]).

(ii) For $r^{-1} = 0$ or $\beta'' = 0$ and $\beta' \neq 0$. Setting $a_{11} = a_{22} = 1$, $a_{12} = v$, $a_{21} = 0$, the following Galilean transformation results:

$$z' = z + vt \qquad t' = t \tag{19}$$

or

$$z = z' - vt' \qquad t = t'.$$

The Galilean invariance states that any solution $\phi(z, t)$ of

$$\mathrm{i}\alpha'\phi_z + \alpha''\phi_{zz} + \mathrm{i}\beta'\phi_t + \kappa\phi + F(|\phi|^2)\phi = 0$$

can be replaced by

$$\phi(z - vt, t) \exp\left\{\mathrm{i}\frac{\beta' v}{2\alpha''}\left[z - \left(\frac{v}{2} + \frac{\alpha'}{\beta'}\right)t\right]\right\}$$

where A and B for θ have been obtained from equations (7) and (9).

The fact that the two forms of the generalized Lorentz transformation (I) and (II) or equations (13) and (14) are independently useful can be further seen as they are applied to obtain the solitary-wave solutions for a special set of N CNKG equations.

For the set of N general CNKG equations given by

$$\mathrm{i}\alpha'_m\phi_{mz} + \alpha''_m\phi_{mzz} + \mathrm{i}\beta'_m\phi_{mt} + \beta''_m\phi_{mtt} + \kappa_m\phi_m + F_m\left(|\phi_1|^2, \ldots, |\phi_N|^2\right)\phi_m = 0 \tag{20}$$

where the subscript $m\,(= 1, \ldots, N)$ denotes the mth component wave, application of our generalized Lorentz transformation requires, for the case $\alpha''_m \neq 0$, $\beta''_m \neq 0$, that $\alpha''_m/\beta''_m = r$ be independent of m, for $m = 1, \ldots, N$. A special set of these N CNKG equations is studied in the following section.

3. Special case of *N* CNKG equations: the L-set

For many applications, the general CNKG equations represented by equation (20) are of the form

$$\mathrm{i}\alpha'_m\phi_{mz} + \alpha''_m\phi_{mzz} + \mathrm{i}\beta'_m\phi_{mt} + \beta''_m\phi_{mtt} + \kappa_m\phi_m + \left(\sum_{j=1}^{N}\lambda_{mj}\left|\phi_j\right|^2\right)\phi_m = 0 \tag{21}$$

where $m = 1, \ldots, N$. For problems of wave propagation in optical fibres in which $\phi_m(z, t)$ represents the amplitude of the mth electric field component, λ_{mj} represents the nonlinear parameter that depends on the nonlinear index coefficient, the effective core area and the carrier frequency ω_m of the mth wave [4]. If $k_{0m} = \omega_m/c$ denotes the wave number at the carrier frequency, $k_{1m} = (\mathrm{d}k/\mathrm{d}\omega)_{\omega=\omega_m}$ denotes $1/v_m$, v_m being the group velocity of the mth wave, and $k_{2m} = (\mathrm{d}^2k/\mathrm{d}\omega^2)_{\omega=\omega_m}$ denotes the group-velocity dispersion (GVD) of the mth wave, then the coefficients of the spatial and time derivatives of ϕ_m in equation (21) can be identified as follows: $\alpha'_m = 1, \alpha''_m = (2k_{0m})^{-1}, \beta'_m = k_{1m}, \beta''_m = -k_{2m}/2$. In most published work that describe the wave propagation in optical fibres, it is usually assumed that the electric field envelope is slowly varying and satisfies the inequality $\left|k_{0m}^{-1}\partial\phi_m/\partial z\right| \ll |\phi_m|$ that allows dropping the $\alpha''_m\phi_{mzz}$ term from equation (21), and that in turn reduces equation (21) to a CNLS equation. However, the inclusion of the $\alpha''_m\phi_{mzz}$ term and consideration of the CNKG rather than CNLS equation not only gives the necessary and important correction to the many solutions of the CNLS equation, but also gives the correct solutions to those cases that involve periodic electric field envelopes with zeroes or those that cross the spatial coordinates at various points, as those presented in [1] and below, for which the inequality is clearly invalid at and near those coordinate points.

In [1], we presented solitary-wave solutions of equation (21) for $N = 1$ and 2 that are applicable for a rather general set of nonlinear interaction parameters λ. The additional solutions for the special set of CNKG equations which we shall present in this section that may have applications in different fields of physics will illustrate the power and usefulness of the generalized Lorentz transformation presented in the previous section.

In analogy with the L-set defined for the special set of N CNLS equations, we define two types of L-set for the N CNKG equations (21). They are given, respectively, by the following (normalized) coupled equations:

$$\text{(I)} \quad \mathrm{i}\alpha'_m\phi_{mz} + \alpha''_m\phi_{mzz} + \mathrm{i}\beta'_m\phi_{mt} \pm \beta''_m\phi_{mtt} + \kappa_m\phi_m \pm \left(\sum_{j=1}^{N}\beta''_m\beta''_j|\phi_j|^2\right)\phi_m = 0 \tag{22}$$

$m = 1, \ldots, N$, where $\beta''_m = +1$ or -1 for $m = 1, \ldots, N$, and $\alpha''_1/\beta''_1 = \ldots = \alpha''_N/\beta''_N \equiv r$; and

$$\text{(II)} \quad \mathrm{i}\alpha'_m\phi_{mz} \pm \alpha''_m\phi_{mzz} + \mathrm{i}\beta'_m\phi_{mt} + \beta''_m\phi_{mtt} + \kappa_m\phi_m \pm \left(\sum_{j=1}^{N}\alpha''_m\alpha''_j|\phi_j|^2\right)\phi_m = 0 \tag{23}$$

$m = 1, \ldots, N$, where $\alpha''_m = +1$ or -1 for $m = 1, \ldots, N$, and $\alpha''_1/\beta''_1 = \ldots = \alpha''_N/\beta''_N \equiv r$.

The special case of equation (22) for N CNLS equations characterized by $\alpha''_m = \beta'_m = 0$ is the L-set for N CNLS equations for which solitary waves in terms of Lamé functions of order $n \leqslant N$ were presented in [1, 6]. As in those works, the standing-wave solutions are first attempted. We then 'boost' these standing-wave solutions to travelling-wave solutions using two forms of generalized Lorentz transformation given in section 2.

For (I), the substitution

$$\phi_m(z, t) = \psi_m(t)\exp[-\mathrm{i}\beta'_m t/(2\beta''_m)]\exp(\mathrm{i}\omega_m z) \tag{24}$$

is made in equation (22), where ω_m are real constants and $\psi_m(t)$ are real functions of t only. Then the coupled equations for $\psi_m(t)$ become

$$\psi_{mtt} + c_m\psi_m + \left(\sum_{j=1}^{N}\beta''_j\psi_j^2\right)\psi_m = 0 \qquad m = 1, \ldots, N \tag{25}$$

where

$$c_m = \pm\beta''_m\left[\kappa_m - \alpha'_m\omega_m - \alpha''_m\omega_m^2 + \beta'^2_m\big/(4\beta''_m)\right]. \tag{26}$$

To eliminate the permutation symmetry, equation (25) is arranged such that

$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_N \tag{27}$$

so that only one of the two choices (the upper or lower sign) in equations (22) and (26) corresponds to the equations of motion for equation (25). Equation (22) is considered with the upper signs since the lower signs give no new physics, and the interaction parameters of equation (22) are characterized by the array $(\beta_1'', \beta_2'', \ldots, \beta_N'')$, where $\beta_j'' = +1$ or -1 (or denoted simply by + or − in [1, 6]), and each of the 2^N arrays is referred to as an interaction type. Tables 3, 4 and appendix D of [1] gave a whole set of analytic solutions for $\psi_m(t)$ in terms of Lamé functions of orders $n \leqslant N$ that are applicable here. The travelling waves are now constructed using the generalized Lorentz invariance (I) which gives the following solitary waves:

$$\psi_m[\eta(t - z/v)] \exp(\mathrm{i}\Phi_m) \tag{28}$$

as a solution of equation (22), where

$$\eta \equiv (1 + rv^{-2})^{-1/2} \tag{29}$$

$$\Phi_m = \left[\eta\left(\omega_m + \frac{\alpha_m'}{2\alpha_m''}\right) - \frac{\alpha_m'}{2\alpha_m''}\right]z + \left[\frac{\eta}{v}\left(r\omega_m + \frac{\alpha_m'}{2\beta_m''}\right) - \frac{\beta_m'}{2\beta_m''}\right]t \tag{30}$$

and where v is the common velocity of the waves.

For $\alpha_m'' = 0$ (i.e. $r = 0$), equations (8) and (9) are used to obtain B and A, giving

$$\Phi_m = \left(\omega_m - \frac{\alpha_m'}{4\beta_m'' v^2}\right)z + \frac{1}{2\beta_m''}\left(\frac{\alpha_m'}{v} - \beta_m'\right)t. \tag{31}$$

For (II), the substitution

$$\phi_m(z, t) = \psi_m(z) \exp[-\mathrm{i}\alpha_m' z/(2\alpha_m'')] \exp(\mathrm{i}\varpi_m t) \tag{32}$$

is made in equation (23), where ϖ_m are real constants and $\psi_m(z)$ are real functions of z only. Then the coupled equations for $\psi_m(z)$ are

$$\psi_{mzz} + c_m\psi_m + \left(\sum_{j=1}^{N} \alpha_j''\psi_j^2\right)\psi_m = 0 \qquad m = 1, \ldots, N \tag{33}$$

where

$$c_m = \pm\alpha_m''[\kappa_m - \beta_m'\varpi_m - \beta_m''\varpi_m^2 + \alpha_m'^2/(4\alpha_m'')]. \tag{34}$$

To eliminate the permutation symmetry, c_j are arranged as in equation (27), and equation (23) with the upper signs only is considered. The interaction parameters of equation (23) are characterized by the array $(\alpha_1'', \alpha_2'', \ldots, \alpha_N'')$, where $\alpha_j'' = +1$ or -1, and each of the 2^N arrays is referred to as an interaction type. The set of analytic solutions for ψ_m given in tables 3, 4 and appendix D of [1] is considered as $\psi_m(z)$ here. The travelling waves are now constructed using the generalized Lorentz invariance (II) which gives the following solitary waves:

$$\psi_m[\sigma(z - vt)] \exp(\mathrm{i}\Psi_m) \tag{35}$$

as a solution of equation (23), where

$$\sigma \equiv (1 + r^{-1}v^2)^{-1/2} \tag{36}$$

$$\Psi_m = \left[\sigma v\left(r^{-1}\varpi_m + \frac{\beta_m'}{2\alpha_m''}\right) - \frac{\alpha_m'}{2\alpha_m''}\right]z + \left[\sigma\left(\varpi_m + \frac{\beta_m'}{2\beta_m''}\right) - \frac{\beta_m'}{2\beta_m''}\right]t \tag{37}$$

and where v is the common velocity of the waves.

For $\beta''_m = 0$ (i.e. $r^{-1} = 0$), equations (7) and (9) are used to obtain A and B, giving

$$\Psi_m = \frac{1}{2\alpha''_m}(\beta'_m v - \alpha'_m)z + \left(\varpi_m - \frac{\beta'_m v^2}{4\alpha''_m}\right)t. \tag{38}$$

Of the 28 analytic pairs of $\psi_m(\tau), m = 1, 2$, presented for $N = 2$ for the L-set of two CNLS equations, 18 are in terms of Lamé functions [7] of order 1 that are simply the Jacobian elliptic functions $sn(\tau)$, $cn(\tau)$, and $dn(\tau)$, of modulus k, where $0 < k^2 \leqslant 1$, and 10 are in terms of Lamé functions of order 2 that involve products of the Jacobian elliptic functions. There are $2n + 1$ Lamé functions $f_j^{(n)}$ of order n that we arrange in the order of descending magnitude of their corresponding eigenvalues, and we number them according to $j = 1, 2, 2', \ldots, (n+1), (n+1)'$. The analytic solution for the pair of $\psi_1(\tau)$ and $\psi_2(\tau)$ given by the pair of Lamé functions $f_j^{(n)}$ and $f_k^{(n)}$ is denoted by $(j, k)_n$. The three Lamé functions of order $n = 1$ and five Lamé functions of order $n = 2$ are

$$f_1^{(1)} = sn(\tau) \qquad f_2^{(1)} = cn(\tau) \qquad f_{2'}^{(1)} = \mathrm{d}n(\tau)$$

and

$$f_{1,3'}^{(2)} = \tfrac{1}{3}(1 + k^2 \mp \sqrt{1 - k^2 + k^4}) - k^2 sn^2(\tau)$$

$$f_2^{(2)} = sn(\tau)cn(\tau) \qquad f_{2'}^{(2)} = sn(\tau)\,\mathrm{d}n(\tau) \qquad f_3^{(2)} = cn(\tau)dn(\tau).$$

For the interaction type characterized by (β''_1, β''_2) or (α''_1, α''_2), where β''_j or α''_j is equal to +1 or −1, the 10 analytic solutions for $(\psi_1(\tau), \psi_2(\tau))$ that consist of Lamé functions of order 2, from [1], are given as follows:

$$\begin{array}{ll} (--): & (1,2)_2,\ (1,2')_2 \\ (-+): & (1,3)_2,\ (1,3')_2,\ (3,3')_2 \\ (+-): & (2,2')_2 \\ (++): & (2,3)_2,\ (2,3')_2,\ (2',3)_2,\ (2',3')_2. \end{array}$$

A simple rule that can be used to explain which combinations of Lamé functions appear as solutions of a given interaction type can be found in [6].

As examples, the generalized Lorentz transformation in two forms (I) and (II) is now used to obtain the travelling solitary-wave solutions for equations (22) and (23) for $N = 2$.

Consider equation (22) for $N = 2$ and the interaction type characterized by $(\beta''_1, \beta''_2) = (+1, +1)$. We assume that $\alpha''_1 = \alpha''_2 > -v^2$. The solitary-wave solution of equation (22) given by the combination of Lamé functions of order 2 represented by the combination $(2, 3)_2$, is

$$\phi_1(z,t) = A_1 sn\{\eta\gamma(t - z/v)\}cn\{\eta\gamma(t - z/v)\}\exp \mathrm{i}\Phi_1$$

$$\phi_2(z,t) = A_2 cn\{\eta\gamma(t - z/v)\}dn\{\eta\gamma(t - z/v)\}\exp \mathrm{i}\Phi_2$$

where η, Φ_1, Φ_2 are given by equations (29) and (30), and where, from [1],

$$A_1^2 = 6k^4\gamma^2 \qquad A_2^2 = 6k^2\gamma^2 \qquad c_1 = (4 - 5k^2)\gamma^2 \qquad c_2 = (1 - 5k^2)\gamma^2$$

and where c_m are given by equation (26) (with the upper sign).

For the 'mixed' interaction type characterized by $(\beta''_1, \beta''_2) = (+1, -1)$, we assume that $\alpha''_1 = -\alpha''_2 > -v^2$. The solution of equation (23) given by the combination of Lamé functions of order 2, $(2, 2')_2$, is

$$\phi_1(z,t) = A_1 sn\{\eta\gamma(t - z/v)\}cn\{\eta\gamma(t - z/v)\}\exp \mathrm{i}\Phi_1$$

$$\phi_2(z,t) = A_2 sn\{\eta\gamma(t - z/v)\}dn\{\eta\gamma(t - z/v)\}\exp \mathrm{i}\Phi_2$$

where, from [1], $A_1^2 = 6k^4k'^{-2}\gamma^2$, $A_2^2 = 6k^2k'^{-2}\gamma^2$, $c_1 = (4 + k^2)\gamma^2$, $c_2 = (1 + 4k^2)\gamma^2$.

Similarly for equation (23), the corresponding travelling solitary-wave solutions for the interaction type characterized by $(\alpha_1'', \alpha_2'') = (+1, +1), (\beta_1'')^{-1} = (\beta_2'')^{-1} < -v^2$, or by $(+1, -1), (\beta_1'')^{-1} = -(\beta_2'')^{-1} < -v^2$, can be obtained by replacing the argument $\eta\gamma(t - z/v)$ of the elliptic functions in the above solutions by $\sigma\gamma(z - vt)$, and the phase Φ by Ψ, where σ and Ψ_m are given by equations (36) and (37), and c_m are given by equation (34).

For the same positive value of $r \equiv \alpha_m''/\beta_m''$, the two forms of corresponding solutions can be shown to coincide, and thus either form can be used. On the other hand, for negative values of r, the two forms of solutions apply to two different regions $r > -v^2$ and $r < -v^2$.

Following these examples, it is now straightforward to apply the entire collection of analytic solutions for the L-set of N CNLS equations, that are expressed in terms of Lamé functions of order $n \leqslant N$ and are given in [1, 6], to the L-sets for N CNKG equations. The appearance of two forms of 'relativistic' factors η and σ given by equations (29) and (36) that give two different solutions for two distinct regions is a novel result that may have other implications.

4. Summary

The following results have been presented:

(1) A generalized Lorentz transformation for a general set of nonlinear Klein–Gordon equations. The (proper) transformation and invariance are shown to consist of two forms that have overlapping as well as two distinct regions of applicability.
(2) The application of the entire sets of analytic solutions in terms of Lamé functions of orders $n \leqslant N$ presented in earlier papers for the L-set of CNLS equations to the two L-sets of N CNKG equations.

As the L-set for N CNLS equations is known to pass the Painlevé test [8, 9], the subset of the two L-sets for N CNKG equations characterized by $\alpha_m''/\beta_m'' = +1$ could be a good candidate for a Painlevé test.

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References

[1] Hioe F T 2003 *J. Phys. A: Math. Gen.* **36** 7307
[2] Hasegawa A and Kodama Y 1995 *Solitons in Optical Communications* (New York: Oxford University Press)
[3] Akhmediev N N and Ankiewicz A 1997 *Solitons* (New York: Chapman and Hall)
[4] Agrawal G P 2001 *Nonlinear Fiber Optics* 3rd edn (New York: Academic)
[5] Lakshmanan M and Rajasekar S 2003 *Nonlinear Dynamics* (Berlin: Springer)
[6] Hioe F T and Salter T S 2002 *J. Phys. A: Math. Gen.* **35** 8913
[7] See e.g. 1955 *Higher Transcendental Functions* vol 3 chapter XV (ed A Erdélyi *et al*) (New York: McGraw-Hill)
[8] Sahadevan R, Tamizhmani K M and Lakshmanan M 1986 *J. Phys. A: Math. Gen.* **19** 1783
[9] Radhakrishnan R, Sahadevan R and Lakshmanan M 1995 *Chaos Solitons Fractals* **5** 2315
Radhakrishnan R and Lakshmanan M 1995 *J. Phys. A: Math. Gen.* **28** 2683